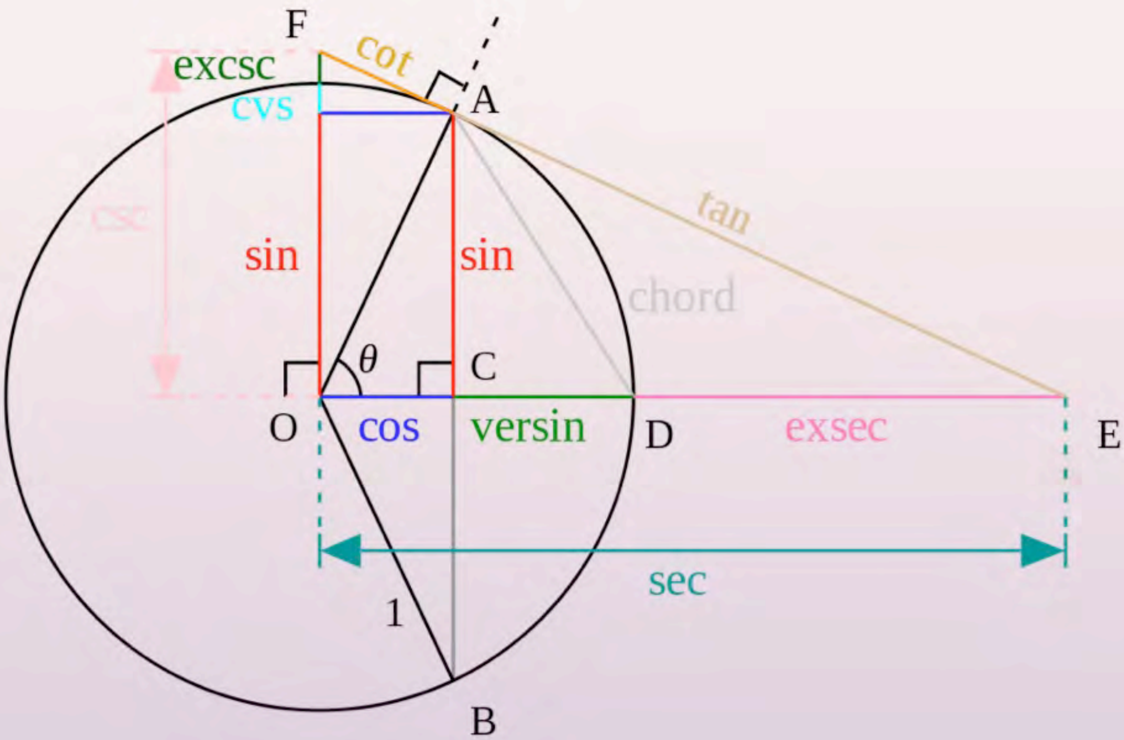


# Trigonometric and exponential derivatives without the diminishing differential

by Michael Howell

Despite the serious problems he identifies in calculus, Miles Mathis upholds the derivatives of sine and cosine (milesmathis.com/trig.html). He demonstrates (or attempts to demonstrate) that their derivatives fall right out of the trigonometric identities.

Sines and cosines are just different measures of $x$ and $y$ on a circle. Their values form the points on the unit circle, whose equation is $r^2 = x^2 + y^2$. Sine and cosine are transcendental functions because their coordinate system is based on radians. A complete revolution, as high-school students learn, is 2π radians (2π times the radius of the unit circle). The arc of a circle is (in general) a transcendental number, though the relation of x to y is simple algebra.

Mathis attempts to prove the derivatives of sine and cosine with simple algebra, but I am afraid he messed up his variables in the last steps. No worry—I can show he has the right premise.

To start the differentiation, let $y = \sin(x) \quad b = \cos(x)$

Remember that, by definition, $\sin^2(x) + \cos^2(x) = 1$

$$z = y^2 = 1 - b^2$$

$$\frac{\Delta z}{\Delta x} = \frac{\Delta\left(y^2\right)}{\Delta x} = \frac{\Delta\left(1 - b^2\right)}{\Delta x}$$

Mathis completely agrees with the standard derivatives of polynomials (based on $x^n$, where $n$ is a positive integer). In fact, he demonstrates how to find the standard derivatives and integrals simply by adding and subtracting in order to form a table of Δ$y$'s and Δ$x$'s (calcsimp.html). That first differential equation we all learn in calculus—

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

—really is, as Mathis asserts, part of the logical definition of integer and exponent:

The table is true by definition. Given the definition of integer and exponent, the table follows. The table is axiomatic number analysis of the simplest kind.

Going back to the derivative of sine, we already know how to differentiate $x$ with respect to $x$—$\Delta(x^2)/\Delta x$, for example, is completely straightforward. But what about $y^2$ or $b^2$ with respect to $x$?

Again, the method falls right out of high-school math:

$$\frac{\Delta\left(y^2\right)}{\Delta x} = \frac{\Delta\left(y^2\right)}{\Delta y} \cdot \frac{\Delta y}{\Delta x} \qquad\qquad \frac{\Delta\left(1 - b^2\right)}{\Delta x} = \frac{\Delta 1}{\Delta x} - \frac{\Delta\left(b^2\right)}{\Delta x} = -\frac{\Delta\left(b^2\right)}{\Delta b} \cdot \frac{\Delta b}{\Delta x}$$

To the credit of standard calculus, the chain rule is commonly demonstrated in the logical, straightforward manner. This simple operation comes out of the definition of fractions and how numerators and denominators can cancel out in multiplication. It is almost hard to believe that the derivatives of polynomials themselves are not taught in such elementary fashion.

Thanks to the useful properties of fractions, we can continue deriving as usual:

$$2y \cdot \frac{\Delta y}{\Delta x} = -2b \cdot \frac{\Delta b}{\Delta x}$$

With the understanding of what $\Delta z/\Delta x$ stands for, let us write in more standard notation:

$$2y \cdot \frac{dy}{dx} = -2b \cdot \frac{db}{dx} \qquad\qquad y \cdot \frac{dy}{dx} = -b \cdot \frac{db}{dx}$$

$$\sin(x) \cdot \frac{dy}{dx} = -\cos(x) \cdot \frac{db}{dx}$$

Solving an equation means finding the substitutions that make both sides look alike. If you have been paying much attention, a bell has probably rung in your head.

$$\frac{dy}{dx} = \cos(x) \qquad\qquad \frac{db}{dx} = -\sin(x)$$

$$\sin(x)\cos(x) = -\cos(x)[-\sin(x)]$$

$$\sin(x)\cos(x) = \cos(x)\sin(x)$$

This differential equation simultaneously proves the derivatives of sine and cosine. It also affirms the standard derivatives for $\sin^2(x)$ and $\cos^2(x)$.

The derivatives for hyperbolic sines are proved in the exact same fashion:

Let $\quad y = \sinh(x) \quad b = \cosh(x)$

By the hyperbolic trigonometric identities, $\quad \cosh^2(x) - \sin^2(x) = 1$

$$z = y^2 = b^2 - 1$$

$$\frac{dz}{dx} = 2y\,\frac{dy}{dx} = 2b\,\frac{db}{dx} \qquad\qquad y\,\frac{dy}{dx} = b\,\frac{db}{dx}$$

$$\sinh(x)\,\frac{dy}{dx} = \cosh(x)\,\frac{db}{dx}$$

This solution should be easier to guess:

$$\frac{dy}{dx} = \cosh(x) \qquad\qquad \frac{db}{dx} = \sinh(x)$$

$$\sinh(x)\cosh(x) = \cosh(x)\sinh(x)$$

In the article “Trigonometric functions,” Wikipedia makes the following statement:

Using only geometry and properties of limits, it can be shown that the derivative of sine is cosine and the derivative of cosine is the negative of sine.

I have already demonstrated that the algebraic relationship of sine and cosine leads to a smashingly simple differential equation. The essence of sine, cosine, and their derivatives has everything to do with the geometry of the circle, and any relation to “limits” is secondary. Wikipedia continues:

One can then use the theory of Taylor series to show that the following identities hold for all real numbers *x*:

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.\end{aligned}$$

These identities are sometimes taken as the *definitions* of the sine and cosine function. They are often used as the starting point in a rigorous treatment of trigonometric functions and their applications (e.g., in Fourier series), since the theory of infinite series can be developed, independent of any geometric considerations, from the foundations of the real number system. The differentiability and continuity of these functions are then established from the series definitions alone.

Combining these two series gives Euler's formula: $\cos x + i \sin x = e^{ix}$.

If one can accomplish a “rigorous treatment of trigonometric functions…independent[ly] of any geometric considerations,” then one can analyze a work of literature without regard for its language. In reality, **math can never be separated from geometry** because **its *raison d’être* is to quantify geometric relations**. It is true that the Taylor series further elaborate on the physical properties of sines and cosines, but their most basic properties are not proven with these series—they are just fleshed out in different notation, corroborating our established definitions in the process.

Continuing with my analysis with differential equations, I will now directly explain the relationship of complex numbers to the circular and hyperbolic sines. By extension, I will demonstrate the fundamental reason the natural base (*e*), coupled with the complex plane, directly links the different conic functions.

The direct link between hyperbolic sine and circular sine is—once again—in the differential equations. We must find a factor that equalizes each pair of “DE’s”.

If $y = \sinh(kx)$,

then the $\Delta x$ for a given $\Delta y$ has been compressed by a factor of $k$. In that case,

$y' = k\cosh(kx)$ and $y'' = k^2 \sinh(kx)$

This application of the chain rule is both graphically and algebraically defended:

$$\frac{df(u(x))}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}$$

By the properties of fractions, the chain rule serves any conceivable function.

I have already confirmed that $[\sin x]' = \cos x$ $[\sin x]'' = -\sin x$

and that $[\sinh x]' = \cosh x$ $[\sinh x]'' = \sinh x$

We want $[\sinh kx]' = \cosh kx$ and $[\sinh kx]'' = -\sinh kx$

Based on the first equation, the necessary $k$-value would first seem to be 1. However, the second equation seems to require $k^2 = -1$. Clearly, we need some extra factor.

The standard relationship between hyperbolic sine and circular sine is

$i\sin x = \sinh ix$ , or $\sin x = -i\sinh ix$

Let's see what happens when we plug this in. (I will explain, in a moment, why "imaginary" numbers divide the way they do.)

$[-i\sinh ix]' = -i \cdot i\cosh ix = -(-1)\cosh ix = \cosh ix$

$[\cosh ix]' = i\sinh ix = -[-i\sinh ix]$

$\therefore [-i\sinh ix]'' = -[-i\sinh ix]$

$\because \sin x = -i\sinh ix$ $[\sin x]'' = -[\sin x]$

$\therefore [\sin x]'' = -\sin x$

Our DE's also confirm that $\cosh ix = \cos x$

Hyperbolic sines and cosines are, respectively, the $y$- and $x$-values of the "unit hyperbola," where $x^2 - y^2 = 1$. Whereas the angles of a circle are based on circumference, the angles of a hyperbola are based on area between a branch of the hyperbola and the nearby asymptote.

As a bonus, hyperbolic sines and cosines can be expressed in closed numerical form. Sines and cosines of a circle, in general, cannot.

$$\sinh x = \tfrac{1}{2}\left(e^{x} - e^{-x}\right) \qquad \cosh x = \tfrac{1}{2}\left(e^{x} + e^{-x}\right)$$

Let’s see what happens when we square each of these terms.

$$\sinh x = \tfrac{1}{2}e^{-x}\left(e^{2x} - 1\right) \qquad \cosh x = \tfrac{1}{2}e^{-x}\left(e^{2x} + 1\right)$$

$$\sinh^2 x = \tfrac{1}{4}e^{-2x}\left(e^{4x} - 2e^{2x} + 1\right) \qquad \cosh^2 x = \tfrac{1}{4}e^{-2x}\left(e^{4x} + 2e^{2x} + 1\right)$$

(Remember the old “FOIL” rule they taught you for multiplying binomials.)

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \tfrac{1}{4}e^{-2x}\left(e^{4x} + 2e^{2x} + 1\right) - \tfrac{1}{4}e^{-2x}\left(e^{4x} - 2e^{2x} + 1\right) \\
&= \tfrac{1}{4}e^{-2x}\left[\left(e^{4x} + 2e^{2x} + 1\right) - \left(e^{4x} - 2e^{2x} + 1\right)\right] \\
&= \tfrac{1}{4}e^{-2x}\left[\underline{e^{4x}} + 2e^{2x} + \underline{\underline{1}} - \underline{e^{4x}} + 2e^{2x} - \underline{\underline{1}}\right] \\
&= \tfrac{1}{4}e^{-2x}\left[2e^{2x} + 2e^{2x}\right] \qquad = \tfrac{1}{4}e^{-2x}\left[4e^{2x}\right] \qquad = e^{-2x}\left[e^{2x}\right] \\
&= 1
\end{aligned}$$

Hyperbolic sine and cosine are, indeed, written in the proper terms for the identity. Also of interest is that when sinh($x$) = 1, cosh($x$) = $\sqrt{2}$ . This is a direct connection with the silver ratio.

It is a short step, now, to prove the standard derivative of $e^x$.

$$\begin{aligned}
\cosh x + \sinh x &= \tfrac{1}{2}e^{-x}\left[\left(e^{2x} + 1\right) + \left(e^{2x} - 1\right)\right] \qquad = \tfrac{1}{2}e^{-x}\left[\left(e^{2x}\right) + \left(e^{2x}\right)\right] \\
&= \tfrac{1}{2}e^{-x}\left[2e^{2x}\right] \qquad = e^{-x}\left[e^{2x}\right] \qquad = e^{x}
\end{aligned}$$

$$\because\ e^{x} = \cosh x + \sinh x$$

$$\left(e^{x}\right)' = \sinh x + \cosh x$$

$$\therefore\ \left(e^{x}\right)' = e^{x}$$

**All** the **properties of the natural base (*e*) come directly from** its role in **forming solutions to particular algebra problems.** The reason ***e* is still transcendental** is that **its *numerical* solution cannot be found *directly* from a finite number of polynomial terms.**

Whereas the **golden mean (*phi*) represents a static equilibrium,** the **natural base represents a *dynamic* equilibrium.** The golden mean comes out of a static algebraic problem, where simple lines are being measured relative to one another. The exponential function comes out of a *dynamic* algebraic problem, where the object of measurement is a *trajectory*.

Similar reasons explain why *pi* [π] is a transcendental number. If *pi* seems a little more absent than the natural base (except for its role in angular coordinates), there is actually a good physical explanation for this, as you will see just below.

I thank Mr. Mathis for teaching me to distinguish between the solution for a *static* problem and the solution for a *kinematic* problem. My work on the stacked spins of the photon (*see my previous three papers, plus* milesmathis.com/elecpro.html) corroborates Mathis' conclusion that *pi* is commonly misused where the appropriate coefficient is 4 (pi2.html).

When the circle is just a line wrapped around a boundary of constant radius (in which case, time does not matter), the proper length is, indeed, 2π. However, when the circle is a series of velocity vectors (which must include time), the proper length is 8. The explanation, again, is simple trigonometry. The trigonometric essentials of Mathis' "Lemma" are summed up here (pi3.html). The easiest way to swallow his revelation is to remember that kinematic solutions demand *parametric* expressions—everything must be a function of $t$ = time, rather than $y$ and $x$.

Here is an excerpt from the long, more detailed version of Mathis' "Lemma":

> [Y]ou cannot directly compare two numbers, when one is a velocity and one is an acceleration…[T]he new number you get from the ratio is not going to be a number that carries any real meaning in it.
>
> …[For] example, what if my acceleration is 3 and your velocity is 1. Can we compare those two numbers directly? No…With an acceleration of 3, my velocity could be anything at a given interval, and my distance traveled likewise. **What if my acceleration is π and your velocity is 1? Is π the value of any real relationship between us? No.** You **can't compare an acceleration to a velocity**. You need more information. … **2πr/t is actually a variable or second-degree acceleration,** of the form **$x/t^3$**. This is because π is already an acceleration itself. This gives 2πr/t the dimensions of $x/t^2/t$, which reduces to $x/t^3$.

Remember that the circumference of **the circle is *always* a product of acceleration**. **Inertia keeps objects** in a **straight** line. Only a radial force pulling inward can keep them in a circular path (or, with gravity, a radial acceleration toward the center, since $g = a$). **A circular path acts as a velocity only when measured in *radians* (based on the "static" circle).**

My proofs for the tangents of curve equations are based on the fact that an object does not keep to a single slope over an interval. Rather, the object travels from the slope at a–1 to that at a+1. Mathis' slope formula, as I contend, is only for static problems, where a curve must be measured as a bunch of discrete lines, rather than a continuous stream of vectors. **The *tangent* is not an average between two points on a line—it is an *average of all the slopes on the interval*. Measured slope must be transformed into true tangent, just as π must sometimes become 4.**

It is well known among mathematicians that 4/π (~1.2732395) approximates the square root of the golden ratio (~1.2720196). The difference between 4 and *pi* seems to explain the difference between the Planck and Wien wavelengths for measuring wavelength of photons from heat.

I take you back to my third paper—"The charge field explains fractals":

Now we find the logarithmic difference (base $e^{1+\sqrt{2}}$) between the fractions we got with the Planck and Wien wavelengths.

$$\frac{\ln(\mathbf{5277.\bar{7}}) - \ln(\mathbf{2944.\bar{4}})}{1+\sqrt{2}} \approx \mathbf{0.241728813}$$

**exp(0.241728813) ≈ 1.2734488**

$$\exp\left[\frac{\ln(\mathbf{5277.\bar{7}}) - \ln(\mathbf{2944.\bar{4}})}{1+\sqrt{2}}\right] \approx \sqrt{\varphi}$$

$\sqrt{\varphi} \approx$ **1.2720196**

My money is on the Planck wavelength. Wikipedia ("Cosmic microwave background radiation") just happens to explain my choice:

The CMBR has a thermal black body spectrum at a temperature of 2.725 K, which peaks at the microwave range frequency of 160.2 GHz, corresponding to a 1.873 mm wavelength. This holds if **measured per unit frequency, as in Planck's law**. If **measured instead per unit wavelength, using Wien's law**, the peak is at 1.06 mm corresponding to a frequency of 283 GHz.

Planck's wavelength seems to be the accurate one for this physical reason: Planck's method is directly connected to time. Wien's measurement divorces its measurements from time.

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As promised, I will now explain "imaginary numbers." Of all the laughable misnomers, this term has to take the cake. The existence of this term is more or less the ultimate proof that Mathis rightly accuses modern math of being jury-rigged with fudges. The numbers themselves, happily enough, are just a natural extension of the number line.

In the study of vectors, students encounter a notation that seems suspiciously like that for complex numbers, except that in addition to **i**, the math has **j** and **k**. That is all a complex number is—a two-dimensional vector. That imaginary numbers are still treated as though something were magical about them seems even more baffling than the way calculus is taught. It is perhaps the ultimate proof that modern mathematicians tend to be out of touch with this world that is the *raison d'être* for the development of math. The king has become oblivious to the people he is supposed to serve.

With two-dimensional vectors, the "real" term acts as the dot product, and the "imaginary term" acts as the cross product. Of course, the true meanings of "dot" and "cross product" cannot be expressed with two dimensions. It takes 4 or 8 dimensions in mathematical space—the first dimension as the domain for the dot product.

The study of vectors began with studies of the behavior of $\sqrt{-1}$. The square root of –1 is not defined for a number line because a line has no intermediate rotations between 0° and 180°. The minus sign is an operator that turns the line segment 180 degrees. Two rotations return the line segment to its original orientation.

It turns out that negative numbers are also "imaginary" in a sense. Originally, only numbers greater than 0 were considered "real" numbers. Being far more obviously connected to the physical world, the thinking of ancient man can often seem *more* practical than modern man's, rather than less. The discovery of negative numbers was kick-started by accounting, where it made intuitive sense to place "in the black" and "in the red" on the same number line.

Wikipedia ("Negative numbers") elaborates once more:

Negative numbers appear for the first time in history in the Nine Chapters on the Mathematical Art (Jiu zhang suan-shu), which in its present form dates from the period of the Han Dynasty (202 BC. – AD 220), but may well contain much older material. The Nine Chapters used red counting rods to denote positive coefficients and black rods for negative. (This system is the exact opposite of contemporary printing of positive and negative numbers in the fields of banking, accounting, and commerce, wherein red numbers denote negative values and black numbers signify positive values). The Chinese were also able to solve simultaneous equations involving negative numbers.

For a long time, negative solutions to problems were considered "false". In Hellenistic Egypt, the Greek mathematician Diophantus in the third century A.D. referred to an equation that was equivalent to $4x + 20 = 0$ (which has a negative solution) in *Arithmetica*, saying that the equation was absurd.

…

During the 7th century AD, negative numbers were used in India to represent debts. The Indian mathematician Brahmagupta, in Brahma-Sphuta-Siddhanta (written in A.D. 628), discussed the use of negative numbers to produce the general form quadratic formula that remains in use today. He also found negative solutions of quadratic equations and gave rules regarding operations involving negative numbers and zero, such as "A debt cut off from nothingness becomes a credit; a credit cut off from nothingness becomes a debt. " He called positive numbers "fortunes," zero "a cipher," and negative numbers "debts."

…

European mathematicians, for the most part, resisted the concept of negative numbers until the 17th century, although Fibonacci allowed negative solutions in financial problems where they could be interpreted as debits (chapter 13 of Liber Abaci, AD 1202) and later as losses (in Flos).

…

In the 18th century it was common practice to ignore any negative results derived from equations, on the assumption that they were meaningless.

There is, once more, good physical reason for the historical resistance to negative numbers. Outside a vector or coordinate system, only numbers greater than 0 have physical meaning. This is why mass is never negative—much less "imaginary"—despite the wishful thinking of some modern physicists. **Physical properties of objects are absolute values. *Negative and complex numbers* (and the number 0) apply only to *positions and motion in space*.**

Wikipedia's article for the number zero sums up the good physical reasons for initial resistance to zero as a concept:

Records show that the ancient Greeks seemed unsure about the status of zero as a number. They asked themselves, "How can nothing be something?", leading to philosophical and, by the Medieval period, religious arguments about the nature and existence of zero and the vacuum. The paradoxes of Zeno of Elea depend in large part on the uncertain interpretation of zero.

Math with three-dimensional vectors began with the study of mathematical space with three orthogonal square roots of –1: *i*, *j*, and *k*. This mathematical space is known as a *quaternion*. For those who always wondered why cross products are not commutative ($\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$), this property comes right out of the behavior of $\sqrt{-1}_{x,y,z}$. Those who could not appreciate why $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ can now observe the mechanical reason. The dot product is simply the "real" component that comes from multiplying the three "imaginary" numbers. (The difference is that modern vector notation just has $\mathbf{i} \cdot \mathbf{i} = 1$.)

Wikipedia has a good article on the "History of quaternions":

From the mid 1880s, quaternions began to be displaced by vector analysis, [whose founders were both] inspired by the quaternions as used in Maxwell's A Treatise on Electricity and Magnetism, but — according to Gibbs — found that "… the idea of the quaternion was quite foreign to the subject." Vector analysis described the same phenomena as quaternions, so it borrowed ideas and terms liberally from the classical quaternion literature. However, vector analysis was conceptually simpler and notationally cleaner, and eventually quaternions were relegated to a minor role in mathematics and physics.

From the main text on "Quaternions":

However, quaternions have had a revival since the late 20th Century, primarily due to their utility in describing spatial rotations. The representations of rotations by quaternions are more compact and quicker to compute than the representations by matrices. In addition, unlike Euler angles they are not susceptible to gimbal lock. For this reason, quaternions are used in computer graphics, computer vision, robotics, control theory, signal processing, attitude control, physics, bioinformatics, molecular dynamics, computer simulations, and orbital mechanics. For example, it is common for the attitude-control systems of spacecraft to be commanded in terms of quaternions. Quaternions have received another boost from number theory because of their relationships with the quadratic forms.

Going back to two-dimensional complex numbers, you may have figured out how to divide by *i*. Since *i* is a rotation by 90 degrees, the reverse is a backward rotation of the same magnitude. That is why $1/i = -i$.