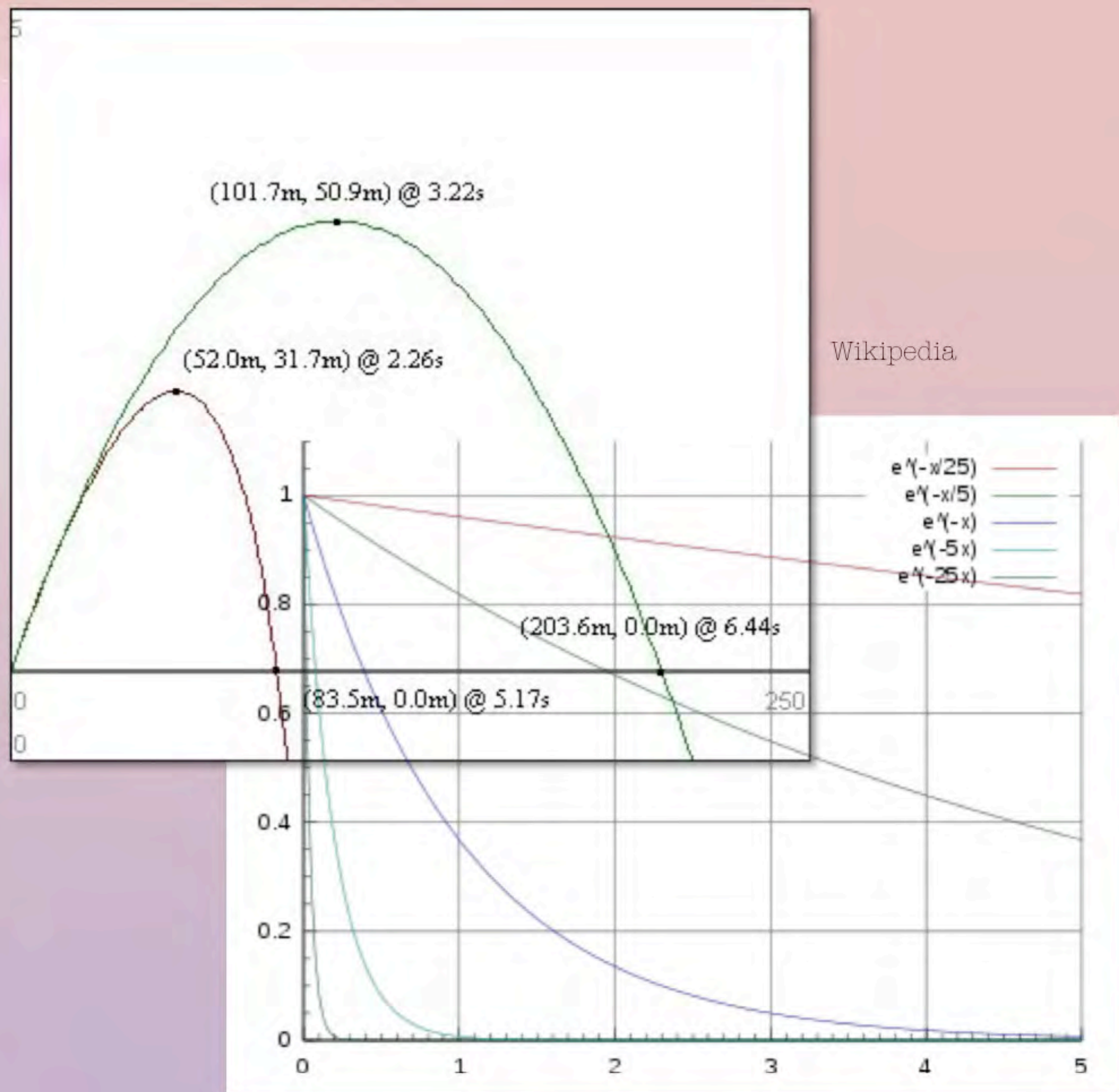

The mechanical cause of exponential decay

by Michael Howell

In addition to the derivative of x^n, the derivatives of ln(x) and e^x form the three centerpieces of differential equations. Differential equations is popularly considered a “post-calculus” math. I have already shown that is entirely backward. Calculus is *based* on differential equations.

Linear differential equations deal primarily with exponential and sinusoidal functions. To call them “linear” seems strange for those unfamiliar with the concept of a differential equation. Wikipedia’s article “Linear differential equation” will (hopefully) clear up the concept.

Perhaps the simplest example of a linear differential equation is the model for radioactive decay:

$$\frac{dN}{dt} = -kN$$

The function $N(t) = e^{-kt}$ has the standard derivative of $N'(t) = -kN(t)$. Linear differential equations are linear in a higher sense of the term than normal linear equations.

Miles Mathis repudiates the diminishing interval for finding slope on a curve. The fact that curvature does not always approach 0 under magnification (see milesmathis.com/expon.html) feels like one serious blow against standard methods. Mathis believes the differential should remain constant—as part of the logical definition of differentiation.

Mathis also challenges the standard derivatives of ln(x) and e^x. He tries to find the slope of an exponential curve the way as for linear equations. He states thusly (milesmathis.com/ln.html):

As with the exponential functions, to find a slope we just find an average…like this:

slope @ (x,y) = [y@(x + 1) - y@(x - 1)]/2

It is true that an exponential curve can never be straightened through normal differentiation. The underlying straight line is the logarithm, by definition. However, Mathis’ operational definition of slope cannot always work for equations with singularities. What Mathis has actually discovered is that some equations can be differentiated in multiple ways. What is appropriately called “the” derivative is that which is most widely useful.

In a static problem, slope has to be measured with a curve of discrete line segments. Measuring change in a meaningful and repeatable way in the real world means finding a reference Δx and sticking with it. When a comet takes a hyperbolic trajectory, the data for displacement relative to time come from a series of individual measurements. Mathis’ slope formula is, indeed, how our data would measure the tangent of the comet’s trajectory.

However, the measured tangent is only an approximation of the true tangent. The static data must be converted to the true behavior of the object. **Although we *measure* a single slope within a given interval, an *object traveling* on the curve *runs through a continuous range of slopes* within that period.** This range of slopes must be summed over and averaged with the appropriate factor. Differentiation and integration really are just two sides of the same coin.

Although contesting the derivatives of exponential and logarithmic functions, Mathis concurs with the derivatives of sine and cosine (milesmathis.com/trig.html). In my piece on trigonometric functions (directly preceding this paper), I demonstrate how trigonometry proves the standard derivative for the exponential function. I repeat the key points down below.

Hyperbolic sines and cosines are, respectively, the *y*- and *x*-values of the "unit hyperbola," where $x^2 - y^2 = 1$. Whereas the angles of a circle are based on circumference, the angles of a hyperbola are based on area between a branch of the hyperbola and the nearby asymptote. As a bonus, *hyperbolic* sines and cosines can be expressed in closed numerical form:

$$\sinh x = \tfrac{1}{2}\left(e^{x} - e^{-x}\right) \qquad \cosh x = \tfrac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \tfrac{1}{4}e^{-2x}\left(e^{4x} + 2e^{2x} + 1\right) - \tfrac{1}{4}e^{-2x}\left(e^{4x} - 2e^{2x} + 1\right)\\
&= \tfrac{1}{4}e^{-2x}\left[\left(e^{4x} + 2e^{2x} + 1\right) - \left(e^{4x} - 2e^{2x} + 1\right)\right]\\
&= \tfrac{1}{4}e^{-2x}\left[\underline{e^{4x}} + 2e^{2x} + \underline{\underline{1}} - \underline{e^{4x}} + 2e^{2x} - \underline{\underline{1}}\right]\\
&= \tfrac{1}{4}e^{-2x}\left[2e^{2x} + 2e^{2x}\right] \qquad = \tfrac{1}{4}e^{-2x}\left[4e^{2x}\right] \qquad = e^{-2x}\left[e^{2x}\right]\\
&= 1
\end{aligned}$$

Hyperbolic sine and cosine are, indeed, written in the proper terms for the identity. Also of interest is that when sinh(*x*) = 1, cosh(*x*) = $\sqrt{2}$. This is a direct connection with the silver ratio.

It is a short step, now, to prove the standard derivative of e^x.

$$\begin{aligned}
\cosh x + \sinh x &= \tfrac{1}{2}e^{-x}\left[\left(e^{2x} + 1\right) + \left(e^{2x} - 1\right)\right] \qquad = \tfrac{1}{2}e^{-x}\left[\left(e^{2x}\right) + \left(e^{2x}\right)\right]\\
&= \tfrac{1}{2}e^{-x}\left[2e^{2x}\right] \qquad = e^{-x}\left[e^{2x}\right] \qquad = e^{x}
\end{aligned}$$

$$\because\ e^{x} = \cosh x + \sinh x$$

$$\left(e^{x}\right)' = \sinh x + \cosh x$$

$$\therefore\ \left(e^{x}\right)' = e^{x}$$

All the **properties of the natural base (*e*) come directly from** its role in **forming solutions to particular algebra problems.** The reason ***e* is still transcendental** is that **its *numerical* solution cannot be found *directly* from a finite number of polynomial terms.**

Whereas the **golden mean (*phi*) represents a static equilibrium,** the **natural base represents a *dynamic* equilibrium.** The golden mean comes out of a static algebraic problem, where simple lines are being measured relative to one another. The exponential function comes out of a *dynamic* algebraic problem, where the object of measurement is a *trajectory*.

The Taylor series of a function is a power series that matches the respective differential equation. Expanding the exponential function into a series is a straightforward process:

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$	$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + O(x^7)$
e^{-x}	$1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720}$

For the series expansion of e^x, the derivative is simply…

$$\frac{d}{dx}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}\right) = \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1 \qquad \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}$$

As per the rules of power derivatives, the derivative of x^k is simply kx^{k-1}. The derivative of 1 is zero. We are not just reducing the power of each term by 1: We are reducing each factorial in the denominator the same way.

Mathis is right to suspect that the methods that work for normal polynomial derivatives are widely abused in finding derivatives of other—often non-algebraic—functions. However, the exponential function can be decomposed into a polynomial. The function is transcendental simply because no algebraic equivalent can be written in *closed* form.

Mathis is also right to conclude that **exponential acceleration** is **physically impossible.** It **can be *approximated* in the physical world, but only the appropriate *fractal* of *power* functions can achieve this.** The universe never could have expanded with true exponential acceleration.

Here are the standard air-resistance equations for the trajectory of a projectile (farside.ph.utexas.edu/teaching/336k/Newton/node29.html):

$$x = \frac{v_0 v_t \cos\theta}{g}\left(1 - e^{-g t/v_t}\right). \qquad z = \frac{v_t}{g}\left(v_0 \sin\theta + v_t\right)\left(1 - e^{-g t/v_t}\right) - v_t t.$$

$$\frac{dv_x}{dt} = -g\,\frac{v_x}{v_t}, \qquad \frac{dv_z}{dt} = -g\left(1 + \frac{v_z}{v_t}\right).$$

$$v_x = v_0 \cos\theta\, e^{-g t/v_t}. \qquad v_z = v_0 \sin\theta\, e^{-g t/v_t} - v_t\left(1 - e^{-g t/v_t}\right).$$

If exponential acceleration is impossible, how do we explain this apparent exception to his rule? Isn't exponential deceleration just acceleration with a minus sign? Again, the exponential curve is a *limit*—not something that is truly reached. Likewise, no physical circle has a circumference of exactly π times its diameter.

Finally, exponential deceleration—look closely—follows a very different power-acceleration pattern. Unlike exponential growth, exponential decay is asymptotic to real-world conditions.

As Mathis has demonstrated before, the charge field interferes with itself. In the third paper of mine that Mathis has published ("The charge field explains fractals," howell3(2).pdf) I use fractals based on the metallic-mean (a.k.a. "silver mean") family to explain absorption bands of plants and photosynthetic bacteria. I now pick up my analysis of fractals where I left off.

The charge field produces the fractal of primary and side effects that would approximate exponential decay of velocity or acceleration. This explains Brownian motion, which Mathis has also covered (milesmathis.com/brown.html). The fractal effects of the charge field contribute to an even more complex fractal that describes the collisions of objects with air particles.

Although air resistance is popularly considered a sum of random particle collisions, the order we see in "chaos" demonstrates that the collisions follow understandable patterns. The problem is that we can never measure anything to infinite precision except objects *defined* as units. The numbers of atoms in macroscopic situations are so great as to make them, for practical purposes, amounts of stuff rather than numbers of items.

Another example of exponential decay in the real world is radioactive decay. This phenomenon is explained in the same basic way as the behavior of air resistance. The only difference is that instead of atomic and molecular particles, the effect comes from subatomic particles. Alpha decay comes from "erosion" by the general charge field. Beta decay comes from collisions of "positrons" (simply upside-down electrons) with neutrons, the aftermath of which creates the illusion of "oscillating neutrinos." In the following papers, Mathis explains how collisions by photons and electrons cause alpha and beta decay:

www.milesmathis.com/nuclear.pdf http://milesmathis.com/quark.html

We already know that the products of beta decay can promote reverse beta decay. The popular interpretation gives credit to the neutrino—rather than the positron (see Wikipedia, "Neutrino Experiment"). This is just one more sign that most physicists are so buried by their math they cannot see the light of day.

***Exponential deceleration* happens for the same reasons as exponential decay of atoms: *collision with a gas of particles*.** Any normal particle gas produces the appropriate fractal. **Exponential growth or decay is the simplest form of "fractal acceleration."**

Newton's law of cooling is also a form of exponential decay:

$$T(t) = T_{\text{env}} + (T(0) - T_{\text{env}})\, e^{-rt}$$

(see Wikipedia, "Convective heat transfer")

Differential equations accomplish the basic work in explaining why *e* really is the "natural" base. Even so, gaining a full appreciation of this number's nature still requires expanding the exponential function into its power series. As the Taylor series demonstrates, **simple exponential growth** stands for the state where **all orders of change equal 1.** All orders of change are perfectly optimized to one another. **All terms in the power series** have an **ultimate derivative of 1.** It is the exponential "identity"—akin to the unit 1.

The **curve of exponential decay** is where **all orders of change alternate between ±1.** This is a sort of net-zero effect. Exponential decay in acceleration is far more likely than exponential growth because the interference pattern is far more natural to generate. Exponential *decay* is a *bounded* process while exponential *growth* is an *unbounded* process. This is why exponential deceleration is pervasive while exponential acceleration is basically unheard of. Basic exponential decay is another sort of exponential "identity"—sharing properties of –1 and 0.

The *raison d'être* of calculus is to straighten out an interval on the curve in order to find the tangent. Power functions of finite terms (normal polynomials) are unique in that a finite number of differentiations bring forth a line of constant value, which is known as the value and order of acceleration. Most functions do not reveal a straight-line relationship in this straightforward a manner, but normal exponentials reveal such a relationship through differential equations.

Specifically, $$\frac{dy}{dt} = k \cdot y(kt)$$

In log-linear view, the exponential graph forms the line $y = kt$. The differential equation above mimics this relationship down to a T. Such a correspondence apparently falls right out of the very meaning of exponential growth. It might not seem, at first, that exponential functions have an algebraic relationship to time; but they, in fact, do.

Through algebraic differentiation, we have finally straightened out the curve. The mathematician just has to think in terms of "dynamic" numbers (*dy/dt*) rather than "static" numbers (*y* and *t*).

The golden ratio and its close relatives are the epitome of static equilibrium. The exponential function is the highest order of equilibrium in a dynamic world. We could already tell this through my differential-equation analysis, but the Taylor series expands on the physics, clarifies them, and corroborates our earlier work.

Down below is the Taylor series for the general exponential function:

$$e^{kx} = 1 + kx + \frac{k^2 x^2}{2} + \frac{k^3 x^3}{6} + \frac{k^4 x^4}{24} + \frac{k^5 x^5}{120} + \frac{k^6 x^6}{720} + O(x^7)$$

Differentiating the second term yields the slope of the log-linear graph. The *n*th-order acceleration in this trajectory equals k^n. Although exponential *acceleration* does not, in general, seem possible to approach, exponential *deceleration* is perfectly easy for nature to approximate. The Taylor series for a negative-exponential function alternates between + and – signs, like so:

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720}$$

As already highlighted, the alternation of signs makes for some sort of "net-zero" effect in the orders of acceleration. Different items produce different log-linear slopes for decay because the exact feedback loops that promote the stable decay rate vary with the object and its interaction with the surrounding particle fluid (liquid, gas, or photons).

Exponential growth is unbounded. It is *not* normally stable. But exponential decay is perfectly bounded. It is a dynamic equilibrium of the "random" particle actions. It is the balance behind the apparent imbalance.

Rarely encountered—but not quite unheard of—is coupled exponential growth, where $y = x^x$. For a proper physical understanding, the more appropriate notation is probably $y = e\text{^}(x \ln x)$. In this case, the decay "constant" depends on time just like the main part of the exponent. This function expands to the following power series:

$$x^x = 1 + x\log(x) + \frac{1}{2}x^2\log^2(x) + \frac{1}{6}x^3\log^3(x) + \frac{1}{24}x^4\log^4(x) + \frac{1}{120}x^5\log^5(x) + \frac{1}{720}x^6\log^6(x) + O(x^7)$$

Like with the other power series, the second term stands for the slope of the log-linear graph. The coupled-exponent graph behaves *quasi*-logarithmically, rather than displaying logarithmic behavior in the familiar sense. For very large values of *x*, ln*x* has a vanishingly small effect relative to *x*, but its effects never become negligible.

The reciprocal of the coupled-exponential function— x^{-x} —mimics the Taylor series for normal exponential decay.

$$1 - x\log(x) + \frac{1}{2}x^2\log^2(x) - \frac{1}{6}x^3\log^3(x) + \frac{1}{24}x^4\log^4(x) - \frac{1}{120}x^5\log^5(x) + O(x^6)$$

Wolfram Alpha presents the steps for finding the derivative of x^x:

Derivative: Hide steps

$$\frac{d}{dx}(x^x) = x^x(\log(x) + 1)$$

Possible derivation:

$$\frac{d}{dx}(x^x)$$

Use the chain rule, $\frac{d}{dx}(x^x) = \frac{du^v}{du}\frac{du}{dx} + \frac{du^v}{dv}\frac{dv}{dx}$, where $u = x$, $v = x$ and $\frac{du^v}{du} = u^{-1+v}\,v$, $\frac{du^v}{dv} = u^v\log(u)$:

$$= x^x\left(\frac{d}{dx}(x)\right) + x^x\log(x)\left(\frac{d}{dx}(x)\right)$$

The derivative of x is 1:

$$= x^x\log(x)\left(\frac{d}{dx}(x)\right) + x^x$$

The derivative of x is 1:

$$= x^x + x^x\log(x)$$

That new term on the outside is the first two terms of the Taylor series for the original function. Here is the derivative of x^{-x}:

$$\frac{d}{dx}(x^{-x}) = -x^{-x}(\log(x) + 1)$$

This imitates the derivative of e^{-x}, in that an extra minus sign occurs in front.

The coupled-exponential function might first seem to be a rare entity, but is well known to engineers who specialize in complex systems. This function and closely related entities are a classic hallmark of turbulent environments, where dynamic equilibrium takes a more complex meaning than generally concerns the layman.

Coupled exponentials, for example, are central to some statistical models of vortex filaments. Google Books[1] offers samples of one paper titled "A Statistical Law of Velocity Circulations in Fully Developed Turbulence" (Yoshida & Hatakeyama). In Section #5—A model of the statistics of vortex filaments—

The authors considered a simplified model of the statistics of velocity circulation caused by a random distribution of vortex filaments in turbulence in [7]. Here we neglect the weaker back ground vorticity field.

I will let the authors describe their model in their own words:

> Let N vortex filaments in which the vorticity is concentrated is randomly distributed in the domain D. We choose a plane S in the domain D. It is assumed that each vortices intersects with S only once and that areas of intersections are small and approximated by points. Let $Y_j \in S (j = 1, \ldots, N)$ and $Z_j \in \mathbf{R} (j = 1, \ldots, N)$ be the intersection point with the plane S and the circulation of the j-th vortex respectively. By assuming the randomness and mutual independence of Y_j and Z_j, it is shown that the circulation in the model denoted by $\Gamma_{\mathrm{m}}(a)$ obeys a compound Poisson distribution $P(\theta a, \sigma)$ whose characteristic function is given by
>
> $$\varphi_{\Gamma_{\mathrm{m}}(a)}(s) = \exp\left(\theta a \int_{-\infty}^{\infty} (e^{\mathrm{i}s\gamma} - 1)\sigma(\gamma)\mathrm{d}\gamma\right), \qquad (1)$$
>
> where $\theta = N/|S|$ is the number density of the intersections and $\sigma(\gamma)$ is the common probability distribution of Z_j.
>
> The tail of a compound Poisson distribution in general decays with the order $\mathsf{P}(|\Gamma_{\mathrm{m}}(a)| \geq x) \sim \exp(-x \log x/c)$ where c depends on σ and it decays slower than that of Gaussian distributions. Normalized PDF of $\Gamma_{\mathrm{m}}(a)$ depends on the area a and approximates Gaussian for large a. We assume the probability distribution σ of the circulation is symmetric. Then it is shown that the second moment $\langle(\Gamma_m(a))^2\rangle$ is proportional to the area.

They have a little more left to say, but this is the pertinent part for our own discussion. The bottom line is the first sentence in that last paragraph:

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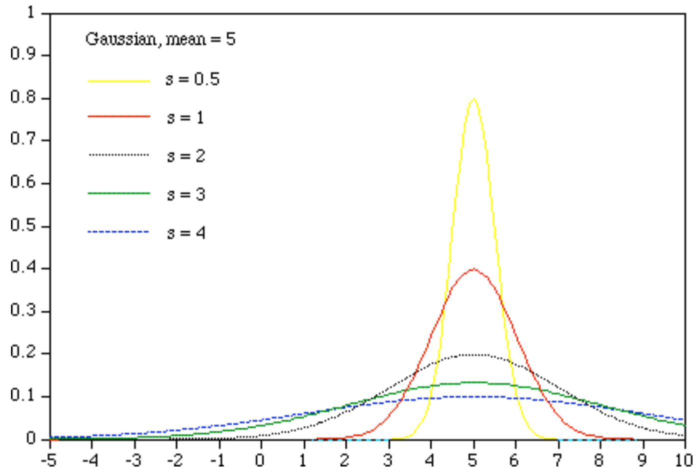
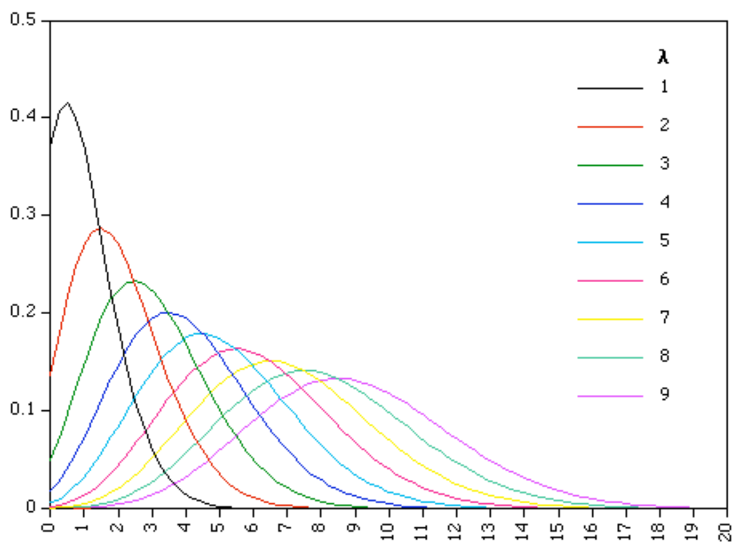
[1] *IUTAM Symposium on Geometry and Statistics of Turbulence* (pp. 145–50)

The **tail of a compound Poisson distribution [P] in general decays** with [**P(| circulation | ≥ *x*) approaching**, for high values of x] **exp(−x log x / c)**, where *c* depends on [probability distribution] σ and it decays slower than that of Gaussian distributions.

Functions like the one above seem unusual only to the layman, who is not a math engineer. For those who are unfamiliar with Poisson distributions, let alone compound ones, here are some graphs for a rudimentary understanding. Let's start[2] by viewing a *normal (Gaussian) distribution*:



That's the old bell curve—used to "normalize" grade curves and IQ tests. Those probability curves that do not fit the "normal" pattern will likely fit the *Poisson distribution*:



Poisson distributions are really three-dimensional functions (at the least); their domain is a plane rather than a line. The proper probability function could very well follow an oblique path to the two axes of the domain (perhaps the gradient).

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[2] http://paulbourke.net/miscellaneous/functions/

It is already known that nuclear decay rates can vary under special conditions. The sun can cause seasonal variations[3], and so can ionization[4]. I recommend a 3D graphical analysis of the exponential decay curve in relation to the possible nuclear decay "constants."
Finally, here is a sample of a *compound Poisson distribution*[5]:

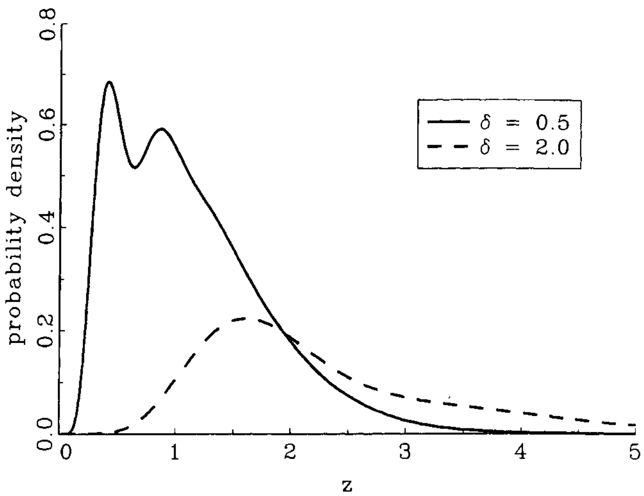
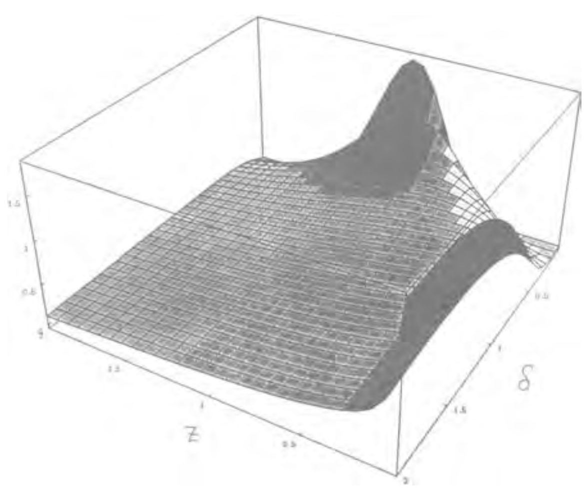


FIG. 5. *Density of the continuous part of the compound Poisson distribution for* $\alpha = 10$, $\gamma = 1$ *and two values of* δ.



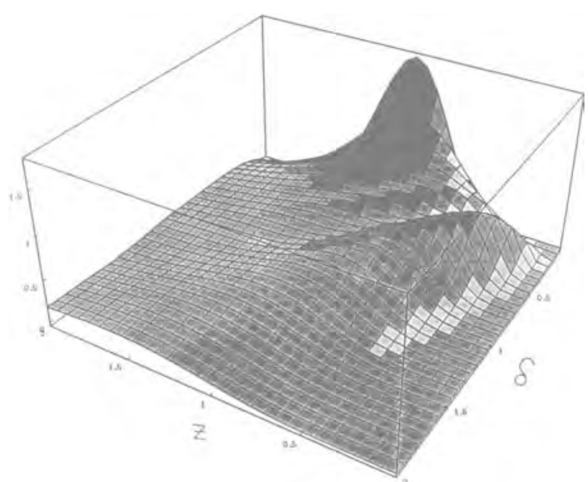


FIG. 1. *Densities of the continuous part of the compound Poisson distribution for* $\alpha = 1.5$. *By varying the parameter* δ *the family of densities is shown as a surface, where a particular density is obtained by cutting the surface parallel to the axis marked* z. *In this case the densities appear unimodal; but see the text for qualifications.* [*Technically, the figure shows* $f(z; 1.5, \delta, 1)$ *for* $0.01 \le z \le 2$ *and* $0.05 \le \delta \le 2$ *with grid size* 0.069×0.067.]

FIG. 4. ***Densities of the continuous part of the compound Poisson distribution for*** $\alpha = 10$; ***see Figure* 1 *for details.***

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[3] http://www.physorg.com/news202456660.html

[4] http://www.aip.org/pnu/1992/split/pnu096-1.htm

[5] Aalen, Odd O. (1992). Modelling heterogeneity in survival analysis by the compound Poisson distribution. Annals of Applied Probability, **2**(4), 951–72.

This function really is compound: it covers four dimensions instead of just three. The domain now fills all the axes of physical space. The ultimate greatness about supercomputers is that they permit, at long last, something resembling a God's-eye view of nature.

The order encountered in "chaos" is ultimate proof that nature is not arbitrary or irrational. Nature simply works in mysterious ways that have to be decoded. Proverbs 25:2 is scientific wisdom as much as it is religious wisdom:

It is the glory of God to conceal a matter; to search out a matter is the glory of kings.

Mathis and I have different creeds; I am orthodox Christian while he is non-religious. But we both agree that nature reveals its secrets only to those who pay their due respects. The more modern physicists have insulted the wisdom of nature, the more their thinking has become futile.

The final section of this treatise is a bonus piece, where I demonstrate a way to pull out the proper derivative of the exponential function—right from Mr. Mathis' slope formula. This experiment was actually what kick-started my whole analysis of the exponential.

The underlying straight line is the key to giving a properly weighted average when finding the slope of an exponential curve. I believe Mathis has revealed that some equations—such as exponential equations—can be differentiated in multiple ways. The differentiation I seek here is that which directly reflects the linearity of $\ln(e^x)$.

For a given function, the slope $f'(x)$ follows this equality:

$$f'(x) = \frac{a\left[f(x+1) - f(x-1)\right]}{K\,\Delta x}$$

$$K\,\Delta x\; f'(x) = a\left[f(x+1) - f(x-1)\right]$$

The a stands for the coefficient in front of x—as in $y = ax$ or $y = e^{ax}$. A coefficient (a) in front of x stretches the run ($\Delta x = 2a$) by a factor of a. An extra a must appear in the numerator for a normalized interval of 2. In the simplest problems, $\Delta x = 2$ ($a = 1$).

K is what I call the "weighting operator." This operator will account for the underlying straight line in the relation between $f(x+1)$ and $f(x-1)$. Strict linear equations will have $K = 1$.

Here is the proof for $y = ax$:

$$f'(x) = \frac{a\left[f(x+1) - f(x-1)\right]}{K\,\Delta x} \qquad = \frac{a\left[(a)(x+1) - (a)(x-1)\right]}{K\,(2a)}$$

$$= \frac{(a)(x+1) - (a)(x-1)}{2K} \qquad = \frac{(ax+a) - (ax-a)}{2K}$$

$$= \frac{ax + a - ax + a}{2K} \qquad = \frac{a+a}{2K} \qquad = \frac{2a}{2K}$$

$$= \frac{a}{K}$$

Of course, the slope for $y = ax$ is simply a. Therefore, $K = 1$.

It is now time to prove that the slope of $y = e^{ax}$ really is $y' = a\,e^{ax}$. Perhaps for the first time, this equality has been proven through keeping Δx constant rather than taking Δx to 0.

$$f'(x) = \frac{a\left[f(x+1) - f(x-1)\right]}{K\,\Delta x} \qquad = \frac{a\left[e^{a(x+1)} - e^{a(x-1)}\right]}{K\,\Delta x}$$

$$K\,\Delta x\; f'(x) = a\left[e^{a(x+1)} - e^{a(x-1)}\right] \qquad K\,\Delta x\; f'(x) = a\left[e^{(ax+a)} - e^{(ax-a)}\right]$$

$$K\ 2a\ f'(x) = a\left[e^{(ax+a)} - e^{(ax-a)}\right] \qquad K\ 2a\ f'(x) = a\left[e^{(ax+a)} - e^{(ax-a)}\right]$$

$$K\ 2a\ f'(x) = ae^{ax}\left[e^{(+a)} - e^{(-a)}\right]$$

$$K\ f'(x) = \frac{ae^{ax}\left[e^{(+a)} - e^{(-a)}\right]}{2a}$$

This is the identity for the hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x}$$

$$\therefore \quad e^a - e^{-a} = 2\sinh a$$

$$K\ f'(x) = \frac{ae^{ax}\left[2\sinh a\right]}{2a} \qquad = \frac{ae^{ax}\left[\sinh a\right]}{a}$$

$$\because \quad K = \frac{\sinh a}{a}$$

$$f'(x) = ae^{ax}$$

Why can an a be canceled out in the simple linear equation—but not the exponential equation? Notice that $y = ax$ produces an a squared.

$$f'(x) = \frac{a\left[(a)(x+1) - (a)(x-1)\right]}{K\,(2a)} \qquad = \frac{a^2\left[(x+1) - (x-1)\right]}{K\,(2a)}$$

This does not happen for $y = e^{ax}$.

$$f'(x) = \frac{ae^{ax}\left[e^{(+a)} - e^{(-a)}\right]}{K\,(2a)}$$

This exercise to uncover hidden linear relationships for the slopes of curves, I believe, has taught me a hidden lesson about operators. Sometimes what looks like a group of discrete terms is, in fact, a whole term that cannot be broken except under restricted conditions.

For the general slope formula, I will repeat a key point I made earlier: *A coefficient (a) in front of x stretches the run (Δx = 2a) by a factor of a. An extra a must appear in the numerator for a normalized interval of 2.* **The 2*a* in the denominator is the reason for putting an *a* into the numerator of the general slope formula.** Only <u>extra</u> *a*'s may be canceled out with 2*a*.

The ***a* in Δ*x* = *2a* belongs to the weighting factor *K*.** Coefficient *K* acts as the transform between simple linear relationships and less direct relationships.