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We Watch Pascal Muck up a Proof

by Miles Mathis

I have shown many of the most famous physicists and mathematicians in history finessing proofs, including Newton, Einstein, Laplace, Lagrange, Feynman, and Maxwell. Today we will look closely at Pascal's proof for the surface area of the sphere, which still stands.

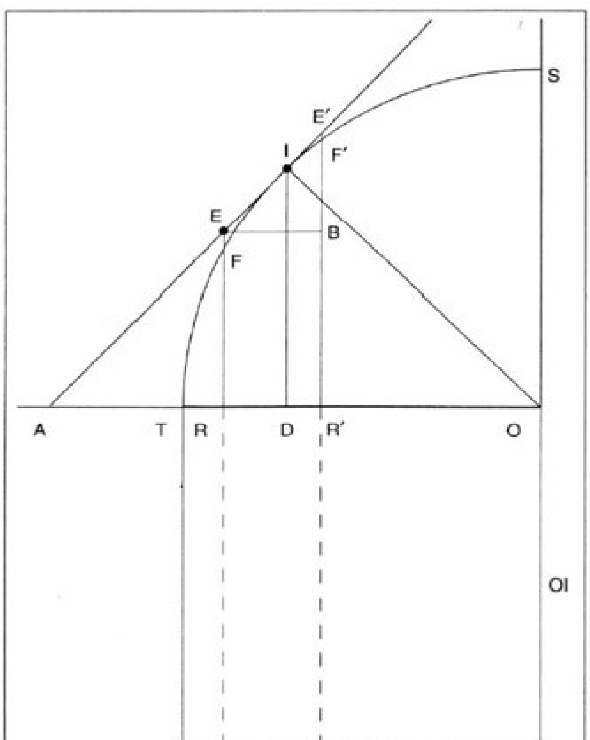


Figure 5

In this figure, OI is a radius. The vertical strip with base RR' is actually of *infinitesimal* width. I is some point located vertically above the width RR'. EB is equal to RR'. EE' is the tangent to the circle at the point I. By the tangent, we mean a line touching the circle at one and only one point. Then we can show that the little infinitesimal triangle, $EE'B$, and the triangle OID are similar. (The line ID divides right triangle AIO into triangles IAD and OID, which are similar to each other, as well as to triangle AIO. That is, they have the same three angles, and therefore their sides are proportional, or, in Leibniz's description, they are indistinguishable apart from their size. $EE'B$ and IAD are similar because their sides are parallel. Because IAD and OID are similar, so are $EE'B$ and OID.)

Based on this similarity of $EE'B$ and IOD, Pascal concluded that $EE' \times DI = RR' \times OI$ (the radius), and that this relationship must hold for each vertical infinitesimal strip! To find the surface for the entire hemisphere, we need the surface generated by rotating the quadrant about the $OR'R$ axis. Each vertical strip or *sinus*, such as $RR'FF'$, when rotated about the base, will generate a circular band upon the hemisphere of arc length FF', that is, an arc length very close to the length of the tangent EE'. Pascal then said, that if we were to take the entire quadrant as divided up into these infinitesimally thin vertical strips, then

$$\Sigma\, EE' \times DI = OI^2,$$

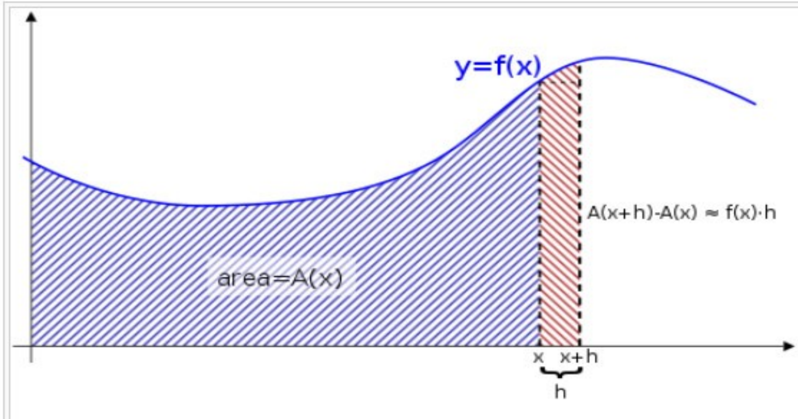
where Σ denotes a process of summation. We get OI^2 on the right side, because OI is being multiplied in succession by each of the lines RR', from O out to T, and their sum is also OI.

But what is the product $EE' \times DI$? It is the area of a cylinder of approximate radius DI and height EE', provided we also multiply by 2π. We say *approximate radius*, because DI lies between the two diameters of the little cylinder, RE' and $R'E'$. The total surface of the hemisphere is obtained by summing up all of these little cylinders. Since the two radii are not exactly equal, that is, RE and $R'E'$, these are not perfect cylinders. This was justified, because as the vertical strip gets thinner and thinner, the tangent line EE' comes closer and closer to being equal to the arc of the circle FF'. Therefore, the area of the infinitesimal cylinder becomes equal to the area of the infinitesimal circular band on the surface of the sphere generated by rotating the quadrant around the axis AO. It gives the result: 2π times the radius squared. Notice that what we were also doing was to construct a rectangle of base equal to the sum of all the RR's and of constant height OI. Because we were summing the RR's all the way out to the end, the rectangle is, in this case, a square. This is illustrated by the strips placed vertically below the line OA. Thus, we have, in fact, been converting the surface of the sphere into a plane area, in this case a square.

This text comes from a 1999 article by Ernest Schapiro. Schapiro is normally quite meticulous, and I am simply assuming he is not making up his own proof. I looked for a copy of Pascal's original proof online but did not find it. If someone wishes to send me a link or copy, I would be happy to receive it. If you can show that these mistakes are Schapiro's and not Pascal's, I will be gratified to hear it, and will add a paragraph here saying so.

This proof is fine until we come to the equation $\Sigma\ EE' \times DI = OI^2$. That equation already kills his proof, because if it is true, then the area of that quadrant is just OI^2. That equation already contains the summation of all DI's. It should be written $\Sigma\ [EE' \times DI] = OI^2$. So the variation in the length of DI across the summation is already included. Which means $\Sigma\ [EE' \times DI] = OI^2$ cannot be true, because OI^2 is the area of a square with side OI. Our quadrant is much smaller than that square. If that equation were true, it would also imply the area of the whole circle was $4OI^2$, which is not what we want regardless. That would imply the area of the circle was the same as the square around it.

I hope you can see that the length EE' is also immaterial, since any other length would do as well. It doesn't even matter that it is a tangent. All that matters is that the length intersects the point I. It is just standing for some width of our strip, which are then taking down to zero. This proof is just like the proof we now use for finding the area under a curve, and in that proof, the width is immaterial. It just has to be a number greater than zero, so that we can sum it into a real number. You can't sum a bunch of zeros into a length. But it doesn't have to be a diagonal, an arc, or a tangent. As you see here:



The width of the strip is just h, or in other diagrams, dx. The length h doesn't have to follow the curve or be a tangent to it, since the height at that point gives us that information.

I will be told, "No, you are mistaking his equation. He isn't finding the area of the quadrant. The summation is not indicating a 2D summation, but a 3D. He is rotating the quadrant about *OR*, as the text says. The equation comes *after* that rotation."

OK, let us see if that makes more sense. That would explain using the length *EE'*, of course, since we need some surface area to sum, don't we? But the 3rd dimension isn't included in that equation at all. Let's study it again.

$\Sigma\ EE'\ x\ DI = OI^2$

The length *EE'* is in the plane of the page, and so are *DI* and *OI*. How are you going to sum those to represent a 360 degree rotation of that plane about *OR*? Shouldn't you have to at least *assign* some length or change in that circle of rotation before you can sum it? You see, this proof is trying to sum *EE'* into that 360 degree rotation, but *EE'* can't sum in that circle since it isn't a length along that circumference of rotation. Since the length *EE'* is in the plane of the page, it can only sum in that direction—in the direction from *T* to *O*. But summing in the direction of *T* to *O* doesn't represent the rotation, as we have seen. Summing from *T* to *O* can only give us the area under the curve—the area of the quadrant. This is why I started my analysis by reminding you of that proof. By reminding you of that other proof, you can see that the equation $\Sigma\ EE'\ x\ DI = OI^2$ doesn't really stand as written. Yes, it leads us in the right direction, but it isn't true.

To say it another way, notice that the proof switches how it is using the length *DI*. At first, *DI* is the height of the strip. But when we go to page 2, suddenly *DI* is the radius of a cylinder (lying on its side). *EE'* is now the height of the cylinder, we are told. Huge problems there, since although those lengths have switched assignments, they are still in the plane of the page. Although we are told that the product *EE' x DI* is a cylinder, it isn't. It is the infinitely thin plane in a *proposed* cylinder. To represent this cylinder, we have to sum that plane 360 degrees. But we can't do that for three reasons. One, you can't sum an infinitely thin plane. Our drawn lines and planes here are not infinitesimally thin, they are infinitely thin, by definition. You might be able to sum an *infinitesimally* thin plane, but to do it you would have to assign some *dx* in the direction of rotation and summation. As I said, we have nothing to sum here, since there is no variable, function, or infinitesimal assigned in the direction of rotation.

Two, the summation sign applies here to a summation along the line *OT*, as the text admits.

We get OI^2 on the right side, because *OI* is being multiplied in succession by each of the lines *RR'*, from *O* out to *T*, and their sum is also *OI*.

As you see, the summation is along *OT*. The Σ is indicating a sum of the cylinders, to create the hemisphere. Therefore, Σ can't also be the sum of infinitesimal planes *EE' x DI* in the 360 degree rotation. The text *proposes* a rotation and cylinders, but nothing in the math represents them. Three, a sphere is 3D, but the equation $\Sigma\ EE' \times DI = OI^2$ is still 2D. That should be clear by the form of the right side. OI^2 is 2D, so the equation cannot be representing the state of affairs after the rotation. As the text says, we have summed down the length *OT* to get OI^2, so where is the third dimension? We need to sum around the circumference of our new cylinder, but that direction of summation is nowhere represented in any of these equations.

I will be told, "It is represented by the 2π. That is what 'sums' your infinitely thin plane into a cylinder." Does it? Well, if that is true, then you are just admitting that the equation $\Sigma\ EE' \times DI = OI^2$ is not 3D. If the number 2π gives us the 3rd dimension all by itself, and if the number 2π is not in that equation, then that equation is not 3D. But if it is not 3D, then it is a false equation, as I have shown above. And you cannot multiply both sides of a false equation by the same number 2π and get a true equation.

You see, what Pascal needs to do in order to sum those cylinders into a sphere is first represent some patch on his arc *RS*. He needs an infinitesimal square, not an infinitesimal length. He needs to rotate that patch 360 degrees to give him an infinitesimal cylinder, then sum that cylinder along the radius *OT*. Done correctly, that might give him the right answer, but that isn't what the math above represents. He needs two summations, one along the radius and the other along the circumference of the cylinder, to represent the three dimensions. But he only has one summation, as you see. And the length *DI* can't help him in the first part of the problem (before the introduction of 2π) because *DI* isn't on the surface. As you see, in the first part, *DI* isn't part of a surface patch, it is part of an interior strip. As such, it can only sum into an interior area, as I showed. If you sum *DI* along *OT*, you can only get an interior area, which cannot translate into a *surface* area.

If you still don't see what I mean, let us study the actual wording of the text even more closely. We are told,

Each vertical strip or *sinus*, such as *RR'FF'*, when rotated about the base, will generate a circular band upon the hemisphere of arc length *FF'*. . . .

Yes, it will *generate* that circular band on the surface, but it doesn't mathematically *represent* that circular band. Each vertical strip is in the form *EE' x DI,* **which is a vertical strip, not a circular band**. In this math, *EE' x DI* is *never* a circular band on the surface of the hemisphere. This should be crystal clear before we multiply both sides by 2π. The length *DI* is interior to the sphere both before and after multiplying by 2π. So *EE' x DI* cannot represent any patch on the surface or circular band on the surface. Therefore, when we sum *EE' x DI* from *O* to *T,* we can only be summing the vertical strips. If we sum the vertical strips, we can only get a 2D summation from *O* to *T*, which is the area of the quadrant. Obviously, multiplying the area of the quadrant by 2π cannot give us the surface area of the hemisphere. Nor can the area of the quadrant be OI^2 or r^2.

What all this means is that the equality cannot be created until after both *DI* and *OI* have been multiplied by 2π. The number 2π physically turns each radius into a circumference, which then allows the math to make sense. But although the first equality $EE' \times DI = RR' \times OI$ is true and very important, the second equality $\Sigma\ EE' \times DI = OI^2$ is simply false as it stands. It is not true until *after* the multiplying by 2π.

You will say, "Well, if the equation is true after multiplying both sides by 2π, then it must be true before. All we have to do is divide both sides of the correct final equation by 2π to get $\Sigma\ EE' \times DI = OI^2$." No, that doesn't work, because the 2π fits into each side differently. On the left side, the 2π goes with *DI* only. We don't multiply the entire summation by 2π. On the right side, the 2π goes with one *OI* but not the other, which is why we get 2π instead of $4\pi^2$. This is how the final equation can be true, but the next to last equation false.

Some will find this critique and analysis caviling, but I don't think it is. For one thing, the first and most important equation $EE' \times DI = RR' \times OI$ was already known as far back as Archimedes. Pascal's only novelty here is developing the proof without referring to Archimedes' cylinder or to his frusta. But Archimedes' assignment of the radius to the outlying cylinder actually helps the proof, making it far clearer. Pascal's assignment of the radius internally like this makes the proof worse, not better. Archimedes also knew how to sum the tiny frusta into a sphere, so Pascal didn't add anything there either. Pascal took something that was already clear and mucked it up, even requiring a bit of a fudge in that last step.

This has been the movement of history since the 17^{th} century, with many problems getting mucked up more and more with each passing century. For while Pascal's proof is only slightly finessed, the current proofs for this problem—using integration—are badly finessed, as we will see in my next paper.